These conditions assure us that the boundary value problem (4.87) has a unique solution. The grid points are given by $x_n = a+nh$, $n = 0(1)N+1$, $h = (b-a)/(N+1)$. A simple difference scheme for (4.87) is written as

$$y_{n-1}-2y_n+y_{n+1} = h^2 f(x_n, y_n, y'_n) \tag{4.88}$$

where the first derivative y_n may be replaced by one of the expressions

$$y'_n = \begin{cases} \text{(i) } (y_{n+1}-y_{n-1})/2h \\ \text{(ii) } (y_n-y_{n-1})/h \\ \text{(iii) } (y_{n+1}-y_n)/h \end{cases} \tag{4.89}$$

The backward and forward differences are accurate to order h. Therefore the difference scheme (4.88) is of $0(h)$. The central difference is accurate to order h^2 and the difference scheme (4.88) will be of $0(h^2)$.

4.4.1 Difference schemes

We now list two difference schemes for the differential Equation (4.87).

Fourth order method

$$\begin{aligned}
\bar{y}'_n &= (y_{n+1}-y_{n-1})/2h \\
\bar{y}'_{n+1} &= (3y_{n+1}-4y_n+y_{n-1})/2h \\
\bar{y}'_{n-1} &= (-y_{n+1}+4y_n-3y_{n-1})/2h \\
\bar{\bar{y}}'_n &= \bar{y}'_n - \frac{h}{20}(\bar{f}_{n+1}-\bar{f}_{n-1}) \\
y_{n-1}-2y_n+y_{n+1} &= \frac{h^2}{12}(\bar{f}_{n+1}+10\bar{\bar{f}}_n+\bar{f}_{n-1})
\end{aligned} \tag{4.90}$$

where $\bar{\bar{f}}_n = f(x_n, y_n, \bar{\bar{y}}'_n)$

and $\bar{f}_{n\pm1} = f(x_{n\pm1}, y_{n\pm1}, \bar{y}'_{n\pm1})$

Sixth order method

$$\begin{aligned}
\bar{y}'_n &= (y_{n+1}-y_{n-1})/2h \\
\bar{y}'_{n+1} &= (3y_{n+1}-4y_n+y_{n-1})/2h \\
\bar{y}'_{n-1} &= (-y_{n+1}+4y_n-3y_{n-1})/2h \\
\bar{\bar{y}}'_{n+1} &= \bar{y}'_n + \frac{h}{3}(2\bar{f}_n+\bar{f}_{n+1}) \\
\bar{\bar{y}}'_{n-1} &= \bar{y}'_n - \frac{h}{3}(2\bar{f}_n+\bar{f}_{n-1}) \\
\bar{y}_{n+1/2} &= \frac{1}{32}(15y_{n+1}+18y_n-y_{n-1})-\frac{h^2}{64}(3\bar{f}_{n+1}+4\bar{f}_n-\bar{f}_{n-1}) \\
\bar{y}_{n-1/2} &= \frac{1}{32}(-y_{n+1}+18y_n+15y_{n-1})-\frac{h^2}{64}(-\bar{f}_{n+1}+4\bar{f}_n+3\bar{f}_{n-1})
\end{aligned}$$

$$\bar{\bar{y}}'_{n+1/2} = \frac{1}{4h}(5y_{n+1}-6y_n+y_{n-1})-\frac{h}{48}(3\bar{f}_{n+1}+8\bar{f}_n+\bar{f}_{n-1})$$

$$\bar{\bar{y}}'_{n-1/2} = \frac{1}{4h}(-y_{n+1}+6y_n-5y_{n-1})+\frac{h}{48}(\bar{f}_{n+1}+8\bar{f}_n+3\bar{f}_{n-1})$$

$$\hat{y}'_n = \bar{y}'_n+h\left[\frac{1}{78}(\bar{f}_{n+1}-\bar{f}_{n-1})-\frac{1}{52}(\bar{\bar{f}}_{n+1}-\bar{\bar{f}}_{n-1}) -\frac{2}{13}(\bar{\bar{f}}_{n+1/2}-\bar{\bar{f}}_{n-1/2})\right]$$

$$y_{n-1}-2y_n+y_{n+1} = \frac{h^2}{60}[26\hat{f}_n+\bar{\bar{f}}_{n+1}+\bar{\bar{f}}_{n-1}+16(\bar{\bar{f}}_{n+1/2}+\bar{\bar{f}}_{n-1/2})] \tag{4.91}$$

where

$$f_n = f(x_n, y_n, \bar{y}'_n),\ \bar{f}_{n\pm1} = f(x_{n\pm1}, y_{n\pm1}, \bar{y}'_{n\pm1})$$

$$\bar{\bar{f}}_{n\pm1} = f(x_{n\pm1}, y_{n\pm1}\ \bar{\bar{y}}'_{n\pm1})$$

$$\bar{\bar{f}}_{n\pm1/2} = f(x_{n\pm1/2}, \bar{y}_{n\pm1/2}, \bar{\bar{y}}'_{n\pm1/2})$$

$$\hat{f}_n = f(x_n, y_n, \hat{y}'_n)$$

4.4.2 Compact implicit difference schemes

These are the implicit relations between the derivatives and the function values at the adjacent nodal points. We use either a Taylor series analysis or a Hermite polynomial interpolation to obtain the relations. We write the difference scheme in the following form

$$\sum_{\nu=-m}^{m}(a_\nu y_{n+\nu}+A_\nu y^{(k)}_{n+\nu}) = 0 \tag{4.92}$$

where $y^{(k)}_{n+\nu}$ represents the kth order derivative of $y(x)$ at $x_{n+\nu}$. The weighting factor a_ν and A_ν are determined by requiring that the method (4.92) satisfies certain accuracy conditions.

We associate with (4.92) the difference operator $L[y(x), h]$ and write

$$L[y(x_n), h] = \sum_{\nu=-m}^{m}[a_\nu y(x_{n+\nu})+A_\nu y^{(k)}(x_{n+\nu})] \tag{4.93}$$

The largest value of p for which the relation

$$L[y(x), h] = 0(h^{k+p}) \tag{4.94}$$

holds for all sufficiently differentiable functions $y(x)$, is the order of the operator. For example, (4.92) and (4.93), for $k = 2$, $m = 1$, become

$$a_{-1}y_{n-1}+a_0 y_n+a_1 y_{n+1}+(A_{-1}y''_{n-1}+A_0 y''_n+A_1 y''_{n+1}) = 0 \tag{4.95}$$

$$L[y(x), h] = a_{-1}y(x_{n-1})+a_0 y(x_n)+a_1 y(x_{n+1}) +A_{-1}y''(x_{n-1})+A_0 y''(x_n)+A_1 y''(x_{n+1}) \tag{4.96}$$

Expanding each term on the right-hand side of (4.96) in the Taylor series about x_n and equating the coefficients of $h^\nu y^{(\nu)}_n/\nu!$, $\nu = 0(1)5$ to zero, we get

$$a_{-1}+a_0+a_1 = 0$$

$$-a_{-1}+a_1 = 0$$

$$a_{-1}+a_1+\frac{2}{h^2}(A_{-1}+A_0+A_1)=0$$

$$-a_{-1}+a_1+\frac{6}{h^2}(-A_{-1}+A_1)=0$$

$$a_{-1}+a_1+\frac{12}{h^2}(A_{-1}+A_1)=0$$

$$-a_{-1}+a_1+\frac{20}{h^2}(-A_{-1}+A_1)=0 \quad (4.97)$$

and

$$L[y(x),h]=\frac{1}{5!}\Bigg[a_{-1}\int_{x_n}^{x_{n-1}}(x_{n-1}-s)^5\,y^{(6)}(s)\,ds+a_1\int_{x_n}^{x_{n+1}}(x_{n+1}-s)^5\,y^{(6)}(s)\,ds$$

$$+20\left(A_{-1}\int_{x_n}^{x_{n-1}}(x_{n-1}-s)^3\,y^{(6)}(s)\,ds+A_1\int_{x_n}^{x_{n+1}}(x_{n+1}-s)^3\,y^{(6)}(s)\,ds\right)\Bigg] \quad (4.98)$$

The last equation in (4.97) is automatically satisfied in view of the second and fourth equations. We are thus left with five equations in six unknowns. We can choose one of the unknowns arbitrary, say $a_1 = 1$, and determine the remaining unknowns. We find

$$a_{-1}=1,\ a_0=-2,\ A_{-1}=A_1=-\frac{1}{12}h^2,\ A_0=-\frac{10}{12}h^2 \quad (4.99)$$

Substituting (4.99) into (4.98) and simplifying we obtain

$$L[y(x),h]=\frac{h^6}{360}\int_{-1}^{1}(1-|u|)^3(3u^2-6|u|-2)y^{(6)}(x_n+hu)\,du$$

$$=-\frac{h^6}{240}\,y^{(6)}(\xi)$$

where $x_n+hu = s$ and $|\xi| < 1$. Equation (4.96) becomes

$$y(x_{n-1})-2y(x_n)+y(x_{n+1})-\frac{h^2}{12}(y''(x_{n-1})+10y''(x_n)+y''(x_{n+1}))=-\frac{n^6}{240}y^{(6)}(\xi)$$

Neglecting the remainder (truncation) term, we get the Numerov method

$$y_{n-1}-2y_n+y_{n+1}=\frac{h^2}{12}(y''_{n-1}+10y''_n+y''_{n+1}) \quad (4.100)$$

which holds good at each nodal point with an error

$$\frac{1}{240}h^6M_6$$

where $M_6=\max\limits_{|\xi|<1}|y^{(6)}(\xi)|$.

From (4.94), we find that the difference equation (4.100) is of order four.

Similarly, for the first order derivative the compact implicit scheme is given by the Milne method

$$y_{n+1}-y_{n-1} = \frac{h}{3}\,(y'_{n+1}+4y'_n+y'_{n-1}) \tag{4.101}$$

For certain type of differential equations, it is useful to replace the derivative term $y^{(k)}$ by the linear differential operator $L[y]$ when constructing the compact implicit difference scheme. We have

$$\sum_{\nu=-m}^{m} (a_\nu\, y_{n+\nu}+A_\nu\,(L[y])_{n+\nu}) = 0 \tag{4.102}$$

For the linear differential operator

$$L[y] = p(x)y''+q(x)y' \tag{4.103}$$

and $m = 1$, we obtain the compact implicit difference scheme

$$q_n^+(L[y])_{n+1}+q_n^0(L[y])_n+q_n^-(L[y])_{n-1}$$
$$= \frac{1}{h^2}\,(r_n^+\, y_{n+1}+r_n^0\, y_n+r_n^-\, y_{n-1}) \tag{4.104}$$

where

$$q_n^+ = 6p_n\, p_{n-1}+h(5p_{n-1}\, q_n-2p_n\, q_{n-1})-h^2\, q_n\, q_{n-1}$$
$$q_n^0 = 4[15\, p_{n+1}\, p_{n-1}-4h(p_{n+1}\, q_{n-1}-q_{n+1}\, p_{n-1})-h^2\, q_{n+1}\, q_{n-1}]$$
$$q_n^- = 6p_n\, p_{n+1}-h(5p_{n+1}\, q_n-2p_n\, q_{n+1})-h^2\, q_n\, q_{n+1}$$
$$r_n^+ = \frac{1}{2}\,[q_n^+(2p_{n+1}+3h\, q_{n+1})+q_n^0(2p_n+hq_n)+q_n^-(2p_{n-1}-hq_{n-1})]$$
$$r_n^- = \frac{1}{2}\,[q_n^+(2p_{n+1}+hq_{n+1})+q_n^0(2p_n-hq_n)+q_n^-(2p_{n-1}-3hq_{n-1})]$$
$$r_n^0 = -(r_n^+ + r_n^-)$$

$p_n = p(x_n)$ and $q_n = q(x_n)$.

However, if we apply the fourth order difference scheme (4.90) to (4.103), we get the difference scheme

$$\gamma_{n+1}\, y_{n+1}+\gamma_n\, y_n+\gamma_{n-1}\, y_{n-1}$$
$$= h^2[l_{n+1}\, r_{n+1}+l_n\, r_n+l_{n-1}\, r_{n-1}] \tag{4.105}$$

where

$$\gamma_{n+1} = p_n\, p_{n-1}\, p_{n+1}+\frac{h}{24}\,(3q_{n+1}\, p_n\, p_{n-1}+10q_n\, p_{n+1}\, p_{n-1}-q_{n-1}\, p_n\, p_{n+1})$$
$$+\frac{h^2}{48}\,(3q_n\, q_{n+1}\, p_{n-1}+q_n\, q_{n-1}\, p_{n+1})$$

$$\gamma_{n-1} = p_n p_{n-1} p_{n+1} + \frac{h}{24}(q_{n+1} p_n p_{n-1} - 10 q_n p_{n+1} p_{n-1} - 3 q_{n-1} p_n p_{n+1})$$

$$+\frac{h^2}{48}(q_n q_{n+1} p_{n-1} + 3 q_n q_{n-1} p_{n+1})$$

$$\gamma_n = -2[p_n p_{n-1} p_{n+1} + \frac{h}{12}(q_{n+1} p_n p_{n-1} - q_{n-1} p_n p_{n+1})$$

$$+\frac{h^2}{24}(q_n q_{n+1} p_{n-1} + q_n q_{n-1} p_{n+1})]$$

$$= -(\gamma_{n+1} + \gamma_{n-1})$$

$$l_{n+1} = \frac{1}{12}\left(p_n p_{n-1} + \frac{h}{2} q_n p_{n-1}\right)$$

$$l_{n-1} = \frac{1}{12}\left(p_n p_{n+1} - \frac{h}{2} q_n p_{n+1}\right)$$

$$l_n = \frac{10}{12} p_{n-1} p_{n+1}$$

$$p_n = p(x_n), \quad q_n = q(x_n) \quad \text{and} \quad r_n = (L[y])_n$$

4.4.3 Difference schemes based on cubic spline function

We shall derive the cubic spline relations which are relevant for constructing the difference schemes for the second order differential equations.

DEFINITION 4.2 A spline function of degree m with nodes at the points $x_n = 0(1)N+1$, is a function $S_\Delta(x)$ with the properties:

(i) On each interval $[x_{n-1}, x_n]$, $n = 1(1)N+1$, $S_\Delta(x)$ is a polynomial of degree m.

(ii) $S_\Delta(x)$ and its first $(m-1)$ derivatives are continuous on $[a, b]$.

If the function $S_\Delta(x)$ has only $(m-k)$ continuous derivatives then k is defined as the deficiency and is usually denoted by $S_\Delta(m, k)$. The cubic spline is a cubic polynomial of deficiency one, i.e. $S_\Delta(3, 1)$. We now use the definition 4.2 to find the cubic spline function approximation for the function $y(x)$, $x \in [a, b]$. We have

$$S''_\Delta(x) = \frac{(x_n - x)}{h} M_{n-1} + \frac{(x - x_{n-1})}{h} M_n \tag{4.106}$$

where primes denote differentiation with respect to x and $S''_\Delta(x_n) = M_n$. Integrating (4.106) and satisfying the interpolating conditions, $S_\Delta(x_{n-1}) = y_{n-1}$ and $S_\Delta(x_n) = y_n$ we obtain the cubic spline approximation function

$$S_\Delta(x) = \frac{(x_n - x)^3}{6h} M_{n-1} + \frac{(x - x_{n-1})^3}{6h} M_n + (y_{n-1} - \frac{h^2}{6} M_{n-1}) \frac{(x_n - x)}{h}$$

$$+ (y_n - \frac{h^2}{6} M_n) \frac{(x - x_{n-1})}{h} \tag{4.107}$$

The function $S_\Delta(x)$ on the interval $[x_n, x_{n+1}]$ is obtained with $n+1$ replacing n in (4.107). The continuity of the first derivative of $S_\Delta(x)$ at $x = x_n$ requires $S'_\Delta(x_n-) = S'_\Delta(x_n+)$. We have

$$\text{(i) } S'_\Delta(x_n-) = \frac{h}{3}M_n+\frac{h}{6}M_{n-1}+\frac{y_n-y_{n-1}}{h}, \quad n = 1(1)N+1$$

$$\text{(ii) } S'_\Delta(x_n+) = -\frac{h}{3}M_n-\frac{h}{6}M_{n+1}+\frac{y_{n+1}-y_n}{h}, \quad n = 0(1)N \tag{4.108}$$

and so that the continuity of the first derivatives implies

$$\frac{h}{6}M_{n-1}+\frac{2h}{3}M_n+\frac{h}{6}M_{n+1} = \frac{1}{h}(y_{n+1}-2y_n+y_{n-1}), \quad n = 1(1)N \tag{4.109}$$

Additional spline relations that are deduciable from (4.107) are listed as follows:

$$\text{(i) } m_n = -\frac{h}{6}(M_{n+1}+2M_n)+\frac{y_{n+1}-y_n}{h}$$

$$\text{(ii) } m_{n+1} = \frac{h}{6}(M_n+2M_{n+1})+\frac{y_{n+1}-y_n}{h}$$

$$\text{(iii) } m_{n+1}-m_n = \frac{h}{2}(M_{n+1}+M_n)$$

$$\text{(iv) } m_{n+1}+m_n = \frac{h}{6}(M_{n+1}-M_n)+\frac{2(y_{n+1}-y_n)}{h} \tag{4.110}$$

The truncation error of the spline functions is obtained by putting $E = e^{hD}$ and expanding in powers of hD. We get the following results.

$$\text{(i) } S'_\Delta(x_n) = m_n = y'(x_n)-\frac{1}{180}h^4y^{(4)}_{(x_n)}+0(h^6)$$

$$\text{(ii) } S''_\Delta(x_n) = M_n = y''(x_n)-\frac{1}{12}h^2y^{(4)}_{(x_n)}+\frac{1}{360}h^4y^{(6)}_{(x_n)}+0(h^6)$$

$$\text{(iii) } S'''_\Delta(x_n) = y'''(x_n)+\frac{1}{2}hy^{(4)}_{(x_n)}+\frac{1}{12}h^2y^{(5)}_{(x_n)}$$

$$-\frac{1}{360}h^4y^{(7)}_{(x_n)}-\frac{1}{1440}h^5y^{(8)}_{(x_n)}+0(h^6) \tag{4.111}$$

From (4.111 iii) we may have

$$\frac{1}{2}(S'''_\Delta(x_n+)+S'''_\Delta(x_n-)) = y'''(x_n)+\frac{1}{12}h^2y^{(5)}_{(x_n)}+0(h^3)$$

$$S'''_\Delta(x_n+)-S'''_\Delta(x_n-) = hy^{(4)}_{(x_n)}-\frac{1}{720}h^5y^{(8)}_{(x_n)}+0(h^7) \tag{4.112}$$

The error $\epsilon(x) = y(x) - S_\Delta(x)$ at any off-nodal point is obtained by substituting (4.111) in the Taylor series expansion of $\epsilon(x_n+\theta h)$, $0 \leqslant \theta \leqslant 1$. We obtain

$$\epsilon(x_n+\theta h) = \frac{\theta^2(\theta-1)^2}{24} h^4 y^{(4)}_{(x_n)} + \frac{\theta(\theta^2-1)(3\theta^2-2)}{360} h^5 y^{(5)}_{(x_n)} + 0(h^6) \quad (4.113)$$

The error is zero for $\theta = 0$ and 1 and also if $y(x)$ is a cubic polynomial so that its fourth and higher derivatives vanish. From (4.113) we get

$$|\epsilon(x_n+\theta h)| \leqslant \max_{0\leqslant\theta\leqslant1} \left\{ \frac{\theta^2(\theta-1)^2}{24} h^4 \,|\, y^{(4)}_{(x_n)} \,| \right\} = \frac{h^4}{384} \,|\, y^{(4)}_{(x_n)} \,| \quad (4.114)$$

Using (4.112) we may write from (4.113) an estimate of the maximum error in $x_n \leqslant x \leqslant x_{n+1}$ as

$$|\epsilon_n| \leqslant \frac{h^3}{384} \max\{|d_n|, |d_{n+1}|\},\ n = 1(1)N-1 \quad (4.115)$$

where

$$y^{(4)}_{(x_n)} = (S'''_\Delta(x_n+) - S'''_\Delta(x_n-))/h + 0(h^4) = \frac{1}{4} d_n + 0(h^4)$$

Now we use the spline function approximation (4.107) to determine difference scheme for the differential equation (4.87). Differentiating $S_\Delta(x)$ in $[x_{n-1}, x_n]$, we get

$$S'_\Delta(x) = M_n \left[\frac{(x-x_{n-1})^2}{2h} - \frac{h}{6}\right] + M_{n-1}\left[-\frac{(x_n-x)^2}{2h} + \frac{h}{6}\right] + \frac{y_n - y_{n-1}}{h} \quad (4.116)$$

Putting $x = x_{n-\lambda} = x_n - \lambda h$ in $S_\Delta(x)$ and (4.116), we have

$$\text{(i)}\ S(x_{n-\lambda}) = M_{n-1}\frac{h^2}{6}\lambda(\lambda^2-1) + M_n\frac{h^2}{6}(1-\lambda)((1-\lambda)^2-1) + \lambda y_{n-1} + (1-\lambda)y_n$$

$$\text{(ii)}\ S'(x_{n-\lambda}) = \frac{y_n - y_{n-1}}{h} - M_{n-1}\frac{h}{2}\left(\lambda^2 - \frac{1}{3}\right) - M_n\frac{h}{2}\left(\frac{1}{3} - (1-\lambda)^2\right) \quad (4.117)$$

where $0 < \lambda \leqslant 1$.

By considering $S_\Delta(x)$ and $S'_\Delta(x)$ in $[x_n, x_{n+1}]$ and putting $x = x_{n+\lambda} = x_n + \lambda h$ we obtain

$$\text{(i)}\ S_\Delta(x_{n+\lambda}) = M_{n+1}\frac{h^2}{6}\lambda(\lambda^2-1) + M_n\frac{h^2}{6}(1-\lambda)((1-\lambda)^2-1) + \lambda y_{n+1} + (1-\lambda)y_n$$

$$\text{(ii)}\ S'_\Delta(x_{n+\lambda}) = \frac{y_{n+1} - y_n}{h} + M_{n+1}\frac{h}{2}\left(\lambda^2 - \frac{1}{3}\right) + M_n\frac{h}{2}\left(\frac{1}{3} - (1-\lambda)^2\right) \quad (4.118)$$

This root is (3, 3) Padé approximation to e^{2R} (see Table 2.4). The graph of ξ, given in Fig. 4.1, has an infinite discontinuity at $R = 2.322\cdots$, which is real positive zero of the denominator $\left(1-R+\frac{2}{5}R^2-\frac{1}{15}R^3\right)$. Beyond this point, ξ is negative and consequently the difference solution is expected to have oscillations when $R \geqslant 2.5$.

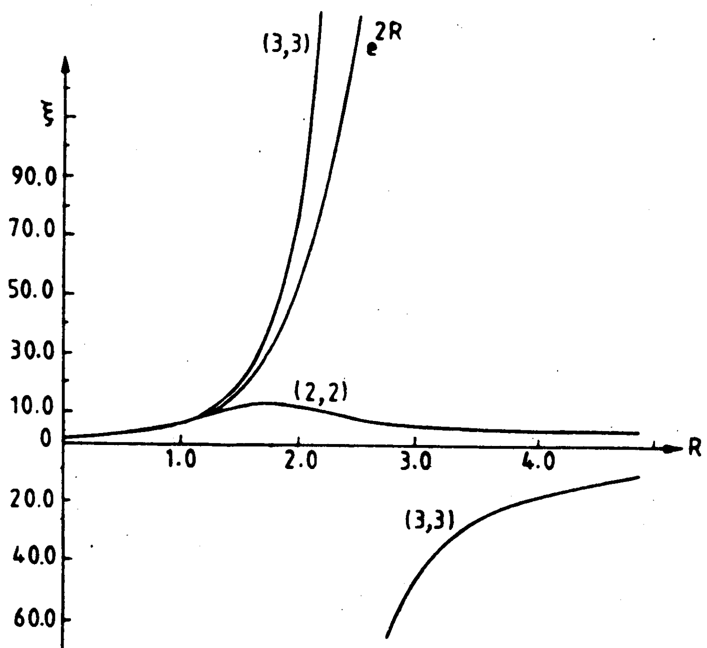

Fig. 4.1 Graph of various order Padé approximations to e^{2R}

Compact implicit difference schemes

Substituting (4.124) into (4.104) we obtain

$$\left(1-R+\frac{1}{3}R^3\right)y_{n+1}-2\,y_n+\left(1+R-\frac{1}{3}R^3\right)y_{n-1}=0 \tag{4.130}$$

The root ξ is given by

$$\xi = \frac{3+R(3-R^2)}{3-R(3-R^2)} \tag{4.131}$$

Three cases are possible for general R:

(i) $R < \sqrt{3}$, $\xi > 1$. The difference solution (4.127) with (4.131) is monotone increasing, concave up, and properly approximates the true solution (4.125).

(ii) $\sqrt{3} < R < 2.1038$ (R value where numerator of ξ vanishes), $0 < \xi < 1$. The difference solution (4.127) is monotone increasing but concave down and completely wrong.
(iii) $R > 2.1038$, $-1 > \xi < 0$. The difference solution is oscillatory.

Compact implicit-block methods

In order to solve the boundary value problem (4.124) with the help of the implicit compact schemes (4.100) and (4.101) we develop the following 3×3 block tridiagonal system of equations

$$\text{(i)} \quad \frac{y_{n+1}-y_{n-1}}{2h} - \frac{m_{n+1}+4m_n+m_{n-1}}{6} = 0$$

$$\text{(ii)} \quad \frac{y_{n+1}-2\,y_n+y_{n-1}}{h^2} - \frac{M_{n+1}+10M_n+M_{n-1}}{12} = 0$$

$$\text{(iii)} \quad M_n - K\,m_n = 0 \tag{4.132}$$

where $m_n \approx (y')_n$ and $M_n \approx (y'')_n$.

The above equations hold for $n = 1\,(1)N$. Alternatively, eliminating M_n and using only y_n, m_n, a block tridiagonal system results from using (4.132i) and

$$\frac{y_{n+1}-2\,y_n+y_{n-1}}{h^2} - \frac{K}{12}(m_{n+1}+10m_n+m_{n-1}) = 0 \tag{4.133}$$

The boundary values ($n = 0,\ N+1$) are required for m_n in (4.133) and for m_n and M_n in (4.132). These are obtained by the methods discussed in Chapter 3. To find the solution of 2×2 system, we write

$$\begin{bmatrix} y_n \\ m_n \end{bmatrix} = \xi^n \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad n = 0(1)\,N+1 \tag{4.134}$$

Substituting (4.134) into (4.132 i) and (4.133), we obtain two homogeneous equations in c_1 and c_2. A nontrivial solution results if the determinental equation

$$(\xi-1)\,[(2-R)\xi^3+(6-11R)\xi^2-(6+11R)\xi-(2+R)] = 0 \tag{4.135}$$

holds. For $R < 2$, there are always three real roots of (4.135), ξ_+, ξ_-, ξ_0 such that

$$\xi_+ > 1,\ \xi_- < -1,\ -1 < \xi_0 < 0$$

A proper analysis of the difference solution will require consideration of the particular schemes used to approximate the required derivatives at the boundaries. However, in the range of R values ($0 < R < 2/(15)^{1/2}$) where $\xi_+ \leqslant |\xi_-|$ no dominant oscillations occur.

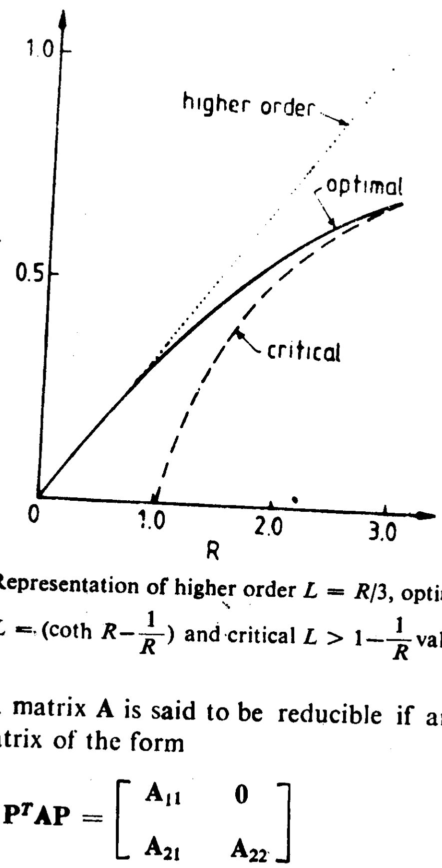

Fig. 4.2 Representation of higher order $L = R/3$, optimal $L = (\coth R - \frac{1}{R})$ and critical $L > 1 - \frac{1}{R}$ values

DEFINITION 4.3 A matrix **A** is said to be reducible if and only if it is similar to a block matrix of the form

$$\mathbf{P}^T\mathbf{AP} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

where $\mathbf{A}_{11}$, $\mathbf{A}_{22}$ are square and **P** is a permutation matrix. In particular, a tridiagonal matrix $\mathbf{A} = (a_{i,j})$ is irreducible if and only if

$$a_{i,i-1} \neq 0 \quad 2 \leqslant i \leqslant n$$

and

$$a_{i,i+1} \neq 0 \quad 1 \leqslant i \leqslant n-1 \tag{4.145}$$

DEFINITION 4.4 A matrix $\mathbf{A} = (a_{i,j})$ is called diagonally dominant if

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} | a_{i,j} | \leqslant | a_{i,i} |, \; 1 \leqslant i \leqslant n \tag{4.146}$$

and strictly diagonally dominant if strict inequality holds in (4.146) for all i; the matrix is irreducibly diagonally dominant if **A** is irreducible, diagonally dominant, and strict inequality holds in (4.146) for at least one i. If by the notation $\mathbf{v} > \mathbf{0}$ (either for vectors or matrices) we mean that all the elements are nonnegative, then we can define: *a matrix* **A** *is monotone if*

$\mathbf{Az} \geqslant 0$ *implies* $\mathbf{z} \geqslant \mathbf{0}$. A direct consequence of the definition is that every monotone matrix is non singular. *A* fundamental result of the theory is:

THEOREM 4.1 *A matrix* $\mathbf{A}$ *is monotone if and only if* $\mathbf{A}^{-1} \geqslant \mathbf{0}$.

Another important result is the following:

THEOREM 4.2 *If a matrix* $\mathbf{A}$ *is irreducibly diagonally dominant and has nonpositive off-diagonal elements then* $\mathbf{A}$ *is monotone.*

Finally we quote for further use:

THEOREM 4.3 *If the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are monotone and* $B \leqslant A$ *then*

$$\mathbf{A}^{-1} \leqslant \mathbf{B}^{-1}$$

We recall the concept of a norm of a vector, $\| \mathbf{x} \|$. The nonnegative quantity $\| \mathbf{x} \|$ is a measure of the size or length of a vector satisfying;

(i) $\| \mathbf{x} \| > 0$, for $\mathbf{x} \neq \mathbf{0}$ and $\| \mathbf{0} \| = 0$

(ii) $\| c\,\mathbf{x} \| = | c | \; \| \mathbf{x} \|$, for an arbitrary complex number c

(iii) $\| \mathbf{x}+\mathbf{y} \| \leqslant \| \mathbf{x} \| + \| \mathbf{y} \|$ (4.147)

We shall in most cases use the maximum norm

$$\| \mathbf{x} \| = \max_{1 \leqslant i \leqslant n} | x_i | \tag{4.148}$$

At this point we must also recall the concept of a matrix norm. In addition to properties analogous to (4.147) the matrix norm must be consistent with the vector norm that we are using for any vector $\mathbf{x}$ and matrix $\mathbf{A}$

$$\| \mathbf{A}\,\mathbf{x} \| \leqslant \| \mathbf{A} \| \; \| \mathbf{x} \|$$

It is easy to verify that the norm

$$\| \mathbf{A} \| = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} | a_{i,j} | \quad \text{(max row sum)} \tag{4.149}$$

is consistent with max norm $\| \mathbf{x} \|$.

The exact solution $y(x)$ of (4.48) satisfies

$$\begin{aligned}(-1+A_n)y(x_{n-1})+(2+B_n)\,y(x_n)+(-1+C_n)\,y(x_{n+1})\\ = D_n - T_n,\ 1 \leqslant n \leqslant N\end{aligned} \tag{4.150}$$

where the truncation error T_n is given by

$$T_n = \frac{1}{12} h^4\, y^{(4)}(\xi_1),\ x_{n-1} < \xi_1 < x_{n+1},\ 1 \leqslant n \leqslant N \tag{4.151}$$

for $\beta_0 = \beta_2 = 0,\ \beta_1 = 1$

and $$T_n = -\frac{1}{240} h^6\, y^{(6)}(\xi_2),\ x_{n-1} \leqslant \xi_2 \leqslant x_{n+1},\ 1 \leqslant n \leqslant N \tag{4.152}$$

where $$f'_M = \max_{a \leqslant x \leqslant b} |f'(x)|$$

Substituting the value of $\| T \|$ in Equation (4.164) from Equation (4.167), we establish the convergence of the sixth order method. In a similar manner, we may prove the convergence of the difference scheme based on the four-point Lobatto quadrature formula as applied to the mixed boundary value problem in Section 4.3.4.

4.6 NONLINEAR BOUNDARY VALUE PROBLEM $y^{(iv)} = f(x, y)$

We consider two-point boundary value problems involving the fourth order differential equation

$$y^{(iv)} = f(x, y) \tag{4.168}$$

With the boundary conditions prescribing either

$$\begin{aligned} y(a) &= A_0, \quad & y(b) &= B_0 \\ y'(a) &= A_1, \quad & y'(b) &= B_1 \end{aligned} \tag{4.169}$$

or

$$\begin{aligned} y(a) &= A_0, \quad & y(b) &= B_0 \\ y''(a) &= A_2, \quad & y''(b) &= B_2 \end{aligned} \tag{4.170}$$

Here, $-\infty < a \leqslant x \leqslant b < \infty$, $A_0, B_0, A_1, B_1, A_2, B_2$ are finite constants. In the following we shall assume that $y(x)$ is sufficiently differentiable and that a unique solution of (4.168) subject to either (4.169) or (4.170) exists.

4.6.1 Difference schemes

Consider the identity

$$\delta^4 y(x_n) = \frac{1}{6}\left\{\int_{x_n}^{x_{n+2}} (x_{n+2}-t)^3 \,[y^{iv}(t)+y^{iv}(2x_n-t)]\, dt - 4\int_{x_n}^{x_{n+1}} (x_{n+1}-t)^3 \,[y^{iv}(t)+y^{iv}(2x_n-t)]\, dt\right\} \tag{4.171}$$

If we use the transformations $t = x_n + h(1+u)$ in the first integral on the right-hand side and $t = x_n + h(1+u)/2$ in the second integral, (4.171) changes into

$$\delta^4 y(x_n) = \frac{h^4}{6}\int_{-1}^{1} (1-u)^3 \Big\{ y^{iv}(x_n - h(1+u)) + y^{iv}(x_n + h(1+u)) - \frac{1}{4} y^{iv}\left(x_n - \frac{h}{2}(1+u)\right) - \frac{1}{4} y^{iv}\left(x_n + \frac{h}{2}(1+u)\right)\Big\} du \tag{4.172}$$

As in Section 4.3.1, we replace the integral in (4.172) with the aid of a suitable quadrature rule and obtain the difference scheme of the form

$$\delta^4 y_n = h^4 \Big\{ W_0 y_n^{iv} + W_1(y_{n-1}^{iv} + y_{n+1}^{iv}) + W_2(y_{n-2}^{iv} + y_{n+2}^{iv}) + \sum_{j=3}^{v} W_j[(y_{n-\theta_j}^{iv} + y_{n+\theta_j}^{iv}) - \frac{1}{4}(y_{n-1/2\theta_j}^{iv} + y_{n+1/2\theta_j}^{iv})] \Big\} \quad (4.173)$$

The values $W_0 = 1$, $W_1 = W_2 = W_j = 0$ give a difference scheme

$$\delta^4 y_n = h^4 y_n^{iv} \quad (4.174)$$

which is of the order two with local truncation error $(1/6)\, h^6\, y^{(6)}(\xi)$, $x_{n-2} < \xi < x_{n+2}$. If we take $W_2 = 0$, $W_j = 0, j = 3, 4 \ldots$, we find that the values $W_0 = 2/3$, $W_1 = 1/6$ give fourth order difference scheme

$$\delta^4 y_n = \frac{h^4}{6}[y_{n-1}^{iv} + 4y_n^{iv} + y_{n+1}^{iv}] \quad (4.175)$$

with the local truncation error

$$T_n^* = -\frac{l}{720} h^8 y^{(8)}(x_n) + \cdots \quad (4.176)$$

In Equation (4.173) if $W_j = 0, j = 3, 4 \ldots$, we can determine W_0, W_1, and W_2 uniquely so that (4.173) has order six. Thus, the sixth order scheme is

$$\delta^4 y_n = \frac{h^4}{720}[474\, y_n^{iv} + 124(y_{n-1}^{iv} + y_{n+1}^{iv}) - (y_{n-2}^{iv} + y_{n+2}^{iv})] \quad (4.177)$$

with local truncation error

$$T_n^* = \frac{1}{3024} h^{10} y^{(10)}(x_n) + \cdots \quad (4.178)$$

We also require the difference expressions for the derivatives $y'(x)$ and $y''(x)$ at the boundary points x_0 and x_{N+1}. We define

$$\text{(i)} \quad \sum_{k=0}^{3} b_k y_k + c h^{1+\alpha} y_0^{(1+\alpha)} + h^4 \sum_{k=0}^{3} d_k f_k = 0$$

$$\text{(ii)} \quad \sum_{k=0}^{3} b_k y_{N+1-k} + c(-h)^{1+\alpha} y_{N+1}^{(1+\alpha)} + h^4 \sum_{k=0}^{3} d_k f_{N+1-k} = 0 \quad (4.179)$$

where $\alpha \in \{0, 1\}$ and b_k, c and d_k, are arbitrary parameters to be determined. Here, we require that the local truncation error in (4.179) to be of the form $0(h^{7+\alpha})$; for then the difference expressions of the boundary conditions (4.169) and (4.170) together with the difference equation (4.177) give $0(h^6)$ difference method. Now so that each difference equation (4.179i) and (4.179ii) is consistent we obtain a system of four equations for the determination of five parameters b_0, b_1, b_2, b_3 and c. If we put $b_3 = 1$, then, for $\alpha = 0$, and $\mathbf{c} = (c_i)$ is the N-dimensional column vector defined by (4.190). The truncation errors associated with equations in (4.189) can be obtained as

$$T_1 = \frac{59}{360} h^6 M_6 \tag{4.193}$$

$$T_n = \frac{1}{6} h^6 M_6, \ 2 \leqslant n \leqslant N-1 \tag{4.194}$$

$$T_N = \frac{59}{360} h^6 M_6 \tag{4.195}$$

where $M_6 = \max_{a \leqslant x \leqslant b} | y^{(6)}(x) |$

We now use the fourth order difference scheme (4.175) to solve the fourth order boundary value problem. We note that the scheme (4.175)

$$\delta^4 y_n = \frac{h^4}{6} (y^{iv}_{n-1} + 4y^{iv}_n + y^{iv}_{n+1}), \ 2 \leqslant n \leqslant N-1 \tag{4.196}$$

gives us $N-2$ relations in the N unknowns y_i, $1 \leqslant i \leqslant N$. If we make use of the boundary conditions (4.186), we get two more relations

$$5y_1 - 4y_2 + y_3 = 2\alpha_1 - h^2\beta_1 + \frac{h^4}{360} (28y^{iv}_0 + 245y^{iv}_1 + 56y^{iv}_2 + y^{iv}_3) \tag{4.197}$$

and

$$y_{N-2} - 4y_{N-1} + 5y_N = 2\alpha_2 - h^2\beta_2 + \frac{h^4}{360} (y^{iv}_{N-2} + 56y^{iv}_{N-1} + 245y^{iv}_N + 28y^{iv}_{N+1}) \tag{4.198}$$

Substituting the values of y^{iv}_n, $0 \leqslant n \leqslant N+1$ in the relations (4.196), (4.197) and (4.198) the system of equations can be replaced by a single matrix equation of the form (4.191),

$$A\mathbf{y} = \mathbf{c}$$

where

$$\mathbf{A} = \begin{bmatrix} 5+\frac{49}{72}h^4f_1 & -4+\frac{7}{45}h^4f_2 & 1+\frac{1}{360}h^4f_3 & & & \\ -4+\frac{1}{6}h^4f_1 & 6+\frac{2}{3}h^4f_2 & -4+\frac{1}{6}h^4f_3 & 1 & & \\ 1 & -4+\frac{1}{6}h^4f_2 & 6+\frac{2}{3}h^4f_3 & -4+\frac{1}{6}h^4f_4 & 1 & \\ \cdot & \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \cdot \\ & & 1 & -4+\frac{1}{6}h^4f_{N-2} & 6+\frac{2}{3}h^2f_{N-1} & 4+\frac{1}{6}h^4f_N \\ & & & 1+\frac{1}{360}h^4f_{N-2} & -4+\frac{7}{45}h^4f_{N-1} & 5+\frac{49}{72}h^4f_N \end{bmatrix}$$

$$\text{and } \mathbf{c} = \begin{bmatrix} \left(2-\frac{7}{90}h^4 f_0\right)\alpha_1 - h^2\beta_1 + \frac{h^4}{360}(28g_0+245g_1+56g_2+g_3) \\ -\alpha_1 \quad + \frac{h^4}{6}(g_1+4g_2+g_3) \\ \vdots \\ -\alpha_2 \quad + \frac{h^4}{6}(g_{N-2}+4g_{N-1}+g_N) \\ \left(2-\frac{7}{20}h^4 f_N\right)\alpha_1 - h^2\beta_2 + \frac{h^4}{360}(g_{N-2}+56g_{N-1}+245g_N+28g_{N+1}) \end{bmatrix}$$

The truncation error of the difference equations is given by

$$|T_n| \leqslant \frac{241}{60480}h^8 M_8, \; n = 1, N$$

$$|T_n| \leqslant 0.002183 h^8 M_8, \; 2 \leqslant n \leqslant N-1$$

where

$$M_n = \max_{a \leqslant x \leqslant b} |y^{(n)}(x)|$$

The matrix **A** is a five-band matrix, the nonzero elements appearing only along the principal diagonals. We can easily extend the method of solution of a tridiagonal system to a five-band system.

4.6.3 Solution of five-band system

The above system of equations can be written as

$$\begin{bmatrix} C_1 & D_1 & E_1 & & & \\ B_2 & C_2 & D_2 & E_2 & & \\ A_3 & B_3 & C_3 & D_3 & E_3 & \\ \vdots & & \cdots\cdots & & & \\ & & A_{N-1} & B_{N-1} & C_{N-1} & D_{N-1} \\ & & & A_N & B_N & C_N \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix} = \begin{bmatrix} \alpha_1^* \\ \alpha_2^* \\ \vdots \\ \alpha_{N-1}^* \\ \alpha_N^* \end{bmatrix} \quad (4.199)$$

where A_i B_i, C_i, D_i, E_i and α_i^* are the known quantities. As in Section 4.3.3, we assume the following recurrence relations

$$y_n = h_n - \omega_n y_{n+1} - \gamma_n y_{n+2}, \; 0 \leqslant n \leqslant N \quad (4.200)$$

We use (4.200) to find y_{n-1} and y_{n-2} and substitute them in the equation

$$A_n y_{n-2} + B_n y_{n-1} + C_n y_n + D_n y_{n+1} + E_n y_{n+2} = \alpha_n^*, \; 2 \leqslant n \leqslant N-2 \quad (4.201)$$

and by comparing it with (4.200), we get

$$h_n = (\alpha_n^* - A_n h_{n-2} - \delta_n h_{n-1})/\omega_n^*$$

$$\omega_n = (D_n - \delta_n \gamma_{n-1})/\omega_n^*$$

$$\gamma_n = E_n/\omega_n^*$$

In fact, (4.208) can be reduced to (4.209) if we assume **B** is nonsingular and $\mathbf{A} = \mathbf{B}^{-1}\mathbf{J}$. Thus, we have reduced (4.205) and (4.206) to the eigenvalue problem (4.209). The eigenvalues and eigenvectors of **A** determined from (4.209) will give approximations to the nontrivial solution of (4.205). We now briefly give some elementary properties of the eigenvalues and eigenvectors of the matrices.

4.7.1 Eigenvalues and eigenvectors

The equations represented by (4.209) are a set of N homogeneous linear equations in N unknowns and such a system of equations will be consistent if and only if

$$\det[\mathbf{A}-\lambda\mathbf{I}] = 0 \tag{4.210}$$

The expansion of this determinant will lead to a polynomial equation of degree $\leqslant N$ in λ. The roots of this are called *eigenvalues* of matrix **A**, and the equation is called the *characteristic equation* of the matrix. The eigenvalues may be distinct or repeated, real or complex. If all the eigenvalues are distinct, there is a nontrivial solution $\mathbf{y}_r$ (eigenvector) corresponding to each eigenvalue λ_r such that

$$\mathbf{A}\,\mathbf{y}_r = \lambda_r\,\mathbf{y}_r \tag{4.211}$$

The eigenvector $\mathbf{y}_r$ is arbitrary to the extent of an indeterminate multiplier. We usually scale the eigenvectors so that they have unit length. This is called *normalizing* the eigenvector. If we premultiply (4.211) by the transpose $\mathbf{y}_r^T$ of $\mathbf{y}_r$, we get

$$\lambda_r = \frac{\mathbf{y}_r^T\mathbf{A}\mathbf{y}_r}{\mathbf{y}_r^T\,\mathbf{y}_r} \tag{4.212}$$

which gives an expression for the eigenvalues in terms of the eigenvectors. For an arbitrary vector **y**, (4.212) is called the *Rayleigh quotient*

$$\lambda_R = \frac{\mathbf{y}^T\,\mathbf{A}\,\mathbf{y}}{\mathbf{y}^T\,\mathbf{y}} \tag{4.213}$$

Let us denote by $\mathbf{A}^T$ the transpose of matrix **A**. Then $(\mathbf{A}^T-\lambda\mathbf{I})$ is the transpose of $(\mathbf{A}-\lambda\mathbf{I})$ and therefore has the same determinant as (4.210). It follows that the characteristic equation and the set of eigenvalues of $\mathbf{A}^T$ are the same as those of **A**. However, the eigenvectors are generally different unless **A** is a symmetric matrix. We can easily prove that

(i) the eigenvalues of a real symmetric matrix are real,
(ii) the eigenvectors of a real symmetric matrix associated with different eigenvalues are orthogonal.

In physical problems we rarely need to determine the whole set of eigenvalues of (4.209). We are generally interested in the *largest* or *smallest* eigenvalue. We assume that the eigenvalues of **A** are real and distinct and we can arrange these as

$$|\lambda_1| > |\lambda_2| \cdots > |\lambda_N| \tag{4.214}$$

Furthermore, let us denote the complete set of eigenvectors of A by $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N$. We can easily determine the largest or smallest eigenvalue and the corresponding eigenvector.

4.7.2 The iteration method

We take an arbitrary initial vector $\mathbf{y}^{(0)}$ which we express as a linear combination of the eigenvectors

$$\mathbf{y}^{(0)} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_N\mathbf{y}_N \tag{4.215}$$

Repeated applications of A give

$$\begin{aligned}\mathbf{y}^{(k)} &= \mathbf{A}\,\mathbf{y}^{(k-1)} = A^k\ \mathbf{y}^{(0)} \\ &= \lambda_1^k\left[c_1\,\mathbf{y}_1 + \sum_{r=2}^{N} c_r\left(\frac{\lambda_r}{\lambda_1}\right)^k \mathbf{y}_r\right]\end{aligned} \tag{4.216}$$

In view of (4.214) and for sufficiently large values of k, the vector

$$c_1\,\mathbf{y}_1 + \sum_{r=2}^{N} c_r\left(\frac{\lambda_r}{\lambda_1}\right)^k \mathbf{y}_r$$

converges to $c_1\mathbf{y}_1$, which is the eigenvector corresponding to the eigenvalue λ_1. The ratio of $\mathbf{y}^{(k+1)}$ to $\mathbf{y}^{(k)}$ will tend to λ_1, that is to say, the ratio of the corresponding elements of $\mathbf{y}^{(k+1)}$ and $\mathbf{y}^{(k)}$ will tend to λ_1. This algorithm is called the *power-method.* From (4.216) the convergence is given by the factor $(\lambda_2/\lambda_1)^k$. In principle this enables us to determine λ_1 and the associated $\mathbf{y}_1$ to any desired accuracy. Unless λ_2/λ_1 is much less than unity, this method is not very efficient. In the case of symmetric matrices, we can obtain better estimates for λ_1 if we use (4.216) to construct the *Rayleigh* quotient (4.213). We obtain

$$\lambda_R = \frac{(\mathbf{y}^{(k)})^T\,(\mathbf{y}^{(k)})}{(\mathbf{y}^{(k)})^T\,(\mathbf{y}^{(k-1)})}$$

For real symmetric matrices, the eigenvectors are orthogonal. Thus we have

$$(\mathbf{y}^{(k)})^T\,(\mathbf{y}^{(k)}) = \sum_{r=1}^{N} c_r^2\lambda_r^{2k} = \lambda_1^{2k}\sum_{r=1}^{N} c_r^2\left(\frac{\lambda_r}{\lambda_1}\right)^{2k}$$

$$(\mathbf{y}^{(k)})^T\,(\mathbf{y}^{(k-1)}) = \sum_{r=1}^{N} c_r^2\lambda_1^{2k-1} = \lambda_1^{2k-1}\sum_{r=1}^{N} c_r^2\left(\frac{\lambda_r}{\lambda_1}\right)^{2k-1}$$

and hence

$$\lambda_R \approx \lambda_1\left[\frac{c_1^2 + \sum_{r=1}^{N} c_r^2\left(\frac{\lambda_r}{\lambda_1}\right)^{2k}}{c_1^2 + \sum_{r=1}^{N} c_r^2\left(\frac{\lambda_r}{\lambda_1}\right)^{2k-1}}\right]$$

The convergence to λ_1 is given by a factor $(\lambda_2/\lambda_1)^{2k}$ which is twice as fast as given by (4.216).

For computation the procedure just described can be formulated in the following way. We use the formula

$$\mathbf{y}^{(k+1)} = \mathbf{A}\mathbf{y}^{(k)}\ k = 0, 1, 2, \ldots \tag{4.217}$$

Thus, the diagonal elements of $\mathbf{B}^{-1}$ are positive. We now determine the following two sums:

$$\text{(i)} \sum_{m=1}^{j-1} | \bar{b}_{jm} | = \frac{3}{\sqrt{6}} \frac{| r^j - ps^j |}{(1-p)} \sum_{m=1}^{j-1} | s^m - r^m |$$

$$\text{(ii)} \sum_{m=j+1}^{N-1} | \bar{b}_{jm} | = \frac{3}{\sqrt{6}} \frac{| s^j - r^j |}{(1-p)} \sum_{m=j+1}^{N-1} | r^m - ps^m | \tag{4.225}$$

Putting $r = -\alpha$ and $s = -\beta$, we obtain

$$\text{(i)} \sum_{m=1}^{j-1} | s^m - r^m | = \frac{1}{8}(\beta^j - \alpha^j) - \frac{1}{8}[(\beta^{j-1} - \alpha^{j-1}) + (\beta - \alpha)]$$

and

$$\text{(ii)} \sum_{m=j+1}^{N-1} | r^m - ps^m | = \frac{1}{8}(\alpha^j - \alpha^{2N}\beta^j) - \frac{1}{8}[\alpha^{j+1}(1 - \alpha^{2(N-j-1)}) + \alpha^{N-1}(1 - \alpha^2)] \tag{4.226}$$

Since the expressions in square brackets in (4.226) are positive, we may write (4.226) as

$$\sum_{m=1}^{j-1} | s^m - r^m | < \frac{1}{8} | s^j - r^j |, \quad j = 2(1)N-1$$

and

$$\sum_{m=j+1}^{N-1} | r^m - ps^m | < \frac{1}{8} | r^j - ps^j |, \quad j = 1(1)N-2 \tag{4.227}$$

Substituting from (4.227) into (4.225) and with $\bar{b}_{jj}$ as given by (4.224), we obtain

$$\text{(i)} \sum_{m=1}^{j-1} | \bar{b}_{jm} | < \frac{1}{8} \bar{b}_{jj}, \quad j = 2(1)N-1$$

and

$$\text{(ii)} \sum_{m=j+1}^{N-1} | \bar{b}_{jm} | < \frac{1}{8} \bar{b}_{jj}, \quad j = 1(1)N-2 \tag{4.228}$$

From (4.228), we have

$$\sum_{\substack{m=1 \\ m \neq j}}^{N-1} | \bar{b}_{jm} | < \frac{1}{4} \bar{b}_{jj}, \quad j = 1(1)N-1$$

which shows that the matrix $\mathbf{B}^{-1}$ is strictly diagonally dominant. We know that the product of two positive definite matrices is a positive definite matrix if and only if the matrices commute. Now, $\mathbf{B}$ is positive definite, the matrix $\mathbf{B}^{-1}$ is positive definite, and it is easily verified that $\mathbf{B}^{-1}$ and $\mathbf{J}$ commute. Thus, the matrix $\mathbf{B}^{-1}\mathbf{J}$ is positive definite. Further, $r(x) \geqslant 0$ on $[a, b]$, therefore $\mathbf{R} > 0$ and hence the matrix $\mathbf{B}^{-1}\mathbf{J}+h^2\mathbf{R}$ is symmetric and positive definite. Thus, the Numerov method gives real and positive approximations for an eigenvalue λ of (4.219). The eigenvalue λ, using the Numerov method is given by

$$(\mathbf{B}^{-1}\mathbf{J}+h^2\mathbf{R})\mathbf{Y} - h^2\lambda\mathbf{Q}\mathbf{Y} = \mathbf{B}^{-1}\mathbf{T} \tag{4.229}$$

where

$$\mathbf{Y} = [y(x_1)y(x_2) \dots y(x_{N-1})]^T,$$
$$\mathbf{T} = [T_1 T_2 \dots T_{N-1}]^T$$
$$T_n = \frac{h^6}{240} y^{(6)}_{(x_n)} + 0(h^8), \quad n = 1(1)N-1 \tag{4.230}$$

We now state the following result.

THEOREM (Keller) 4.6 For each eigenvalue λ of (4.219) and corresponding normalized eigenvector $\mathbf{Y}(x)$, there exists an eigenvalue $h^2 \Lambda$ of $\mathbf{Q}^{-1}(\mathbf{B}^{-1}\mathbf{J} + h^2 \mathbf{R})$ such that

$$|\Lambda - \lambda| \leqslant \| Q^{-1} \| \quad \| \mathbf{B}^{-1} \| \ \frac{\| \boldsymbol{\tau}(\mathbf{Y}) \|}{\| \mathbf{Y} \|} \tag{4.231}$$

where $h^2 \boldsymbol{\tau} = \mathbf{T}$.

From this result we obtain the error estimates in the maximum norm $\| \boldsymbol{\tau}(\mathbf{Y}) \|_\infty$. We use the normalization

$$\int_a^b q(x) y^2(x)\, dx = 1$$

We have

(i) $$h \| \mathbf{Y} \|^2 = \left[\sum_{j=1}^{N-1} h y^2(x_j) + \frac{h}{2}(y^2(x_0) + y^2(x_N)) - \int_a^b y^2(x)\, dx + \int_a^b y^2(x)\, dx \right]$$
$$= \int_a^b y^2(x)\, dx + \frac{h^2}{12} y''(c), \ c \in [a, b]$$
$$\geqslant \frac{1}{q^*} + \frac{h^2}{12} y''(c)$$

where $q^* = \max_{a \leqslant x \leqslant b} q(x)$.

(ii) $h \| \boldsymbol{\tau} \|^2 = h \sum_{j=1}^{N-1} \tau_j^2 \leqslant (b-a) \| \boldsymbol{\tau} \|_\infty^2$

(iii) $\| \boldsymbol{\tau} \| = 0(h^4)$

(iv) $\| \mathbf{B}^{-1} \| = \frac{3}{2}$

(v) $| \mathbf{Q}^{-1} \| = \frac{1}{q_*}$

where

$$q_* = \min_{a \leqslant x \leqslant b} q(x) \tag{4.232}$$

Substituting from (4.232) we may write (4.231) as

$$|\Lambda - \lambda| \leqslant 0(h)^4 \tag{4.233}$$

Thus, we obtain that as $h \to 0$ any fixed eigenvalue λ of (4.219) is approximated by some eigenvalue of the difference equation (4.220) with an error of $0(h^4)$.

TABLE 4.7 COMPARISON OF ERRORS IN NUMERICAL METHODS FOR THE MIXED BOUNDARY VALUE PROBLEM
$y''-y+4xe^x = 0,\ y'(0)-y(0) = 1,\ y'(1)+y(1) = -e$ WITH $h = 2^{-m}$

m	Second order method	Fourth order method	*Approximation* I		*Approximation* II	
			Lobatto method	Gauss method	Lobatto method	Gauss method
2	0.807−01	0.364−03	0.379−06	0.309−06	0.174−05	0.174−05
3	0.203−01	0.232−04	0.600−08	0.482−08	0.304−07	0.304−07
4	0.509−02	0.146−05	0.941−10	0.756−10	0.502−09	0.501−09
5	0.127−02	0.913−07	0.147−11	0.118−11	0.806−11	0.806−11
6	0.319−03	0.571−08	0.238−13	0.183−13	0.127−12	0.128−12
7	0.797−04	0.357−09	0.567−15	0.243−15	0.177−14	0'196−14
8	0.199−04	0.212−10	0.795−16	0.103−16	0.455−16	0.247−16

TABLE 4.8 COMPARISON OF ERROR IN SIXTH ORDER METHOD FOR THE NONLINEAR BOUNDARY VALUE PROBLEMS WITH AND WITHOUT MIXED BOUNDARY CONDITIONS WITH $h = 2^{-m}$

m	$y'' = \frac{1}{2}(1+x+y)^3$		$y'' = \frac{3}{2}y^2$	
	$y(0) = y(1) = 0$	$y'(0)-y(0) = -\frac{1}{2}$, $y'(1)+y(1) = 1$	$y(0) = 4$, $y(1) = 1$	$y'(0)-y(0) = -12$, $y'(1)+y(1) = 0$
3	0.270−06	0.629−06	0.488−05	0.930−05
4	0.435−08	0.125−07	0.797−07	0.187−06
5	0.718−10	0.290−09	0.126−08	0.339−08
6	0.432−11	0.651−10	0.204−10	0.628−10

These results are quite reasonable since in this case $y(x) < 10^{-8}$ for $x > 9$. Thus, we find that the position of the finite points depends on ϵ and to some extent on h also.

The nonlinear differential equations with or without mixed boundary conditions have been solved with $h = 2^{-m}$, $3 \leqslant m \leqslant 6$. The results obtained with the four-point *Lobatto* quadrature with *Approximation* II are listed in Table 4.8. We find that the sixth order methods with *Approximation* II are particularly useful for nonlinear differential equations with or without mixed boundary conditions since we need to solve a fewer number of nonlinear equations to get higher accurate results.

Finally, we arrive at the following conclusions:

(i) The sixth order methods with *Approximation* II are applicable to linear and nonlinear differential equations with or without mixed boundary conditions.

(ii) The numerical results show that the sixth order method based on four-point *Lobatto* quadrature and *Approximation* II is highly desirable for both linear and nonlinear boundary value problems.

4.9 NONUNIFORM GRID METHODS FOR THE SECOND ORDER BOUNDARY VALUE PROBLEMS

Let $a = x_0 < x_1 < x_2, \cdots, x_{N-1} < x_N = b$ be a subdivision of an interval $[a, b]$, where $h_j = x_j - x_{j-1}$, $j = 1\,(1)\,N$. We now obtain the difference schemes for the second order differential equations which when used to solve the two point boundary value problem will lead to a tridiagonal system.

4.9.1 Nonlinear boundary value problems $y'' = f(x, y)$

Let us approximate the differential equation $y'' = f(x, y)$ by the difference scheme of the form

$$2y_n - C_{1n}y_{n-1} - C_{2n}y_{n+1} + h_{n+1}^2 (B_{0n}f_{n-1} + B_{1n}f_n + B_{2n}f_{n+1}) = 0 \qquad (4.234)$$

where f_n and y_n represent the approximate values of $f(x_n, y(x_n))$ and $y(x_n)$, respectively. The C's and B's are unknowns to be determined. We now write the difference operator $L[y(x), h_n]$ associated with the equation (4.234) as

$$L[y(x), h_n] = 2y(x_n) - C_{1n}y(x_n - h_n) - C_{2n}y(x_n + h_{n+1}) + h_{n+1}^2 [B_{0n}y''(x_n - h_n) + B_{1n}y''(x_n) + B_{2n}y''(x_n + h_{n+1})] \qquad (4.235)$$

We expand the various y's on the right-hand side of (4.235) in the Taylor series about x_n and equate the coefficients of $h_n^{\nu} y_{(x_n)}^{(\nu)}$, $(\nu = 0\,(1)\,4)$ to zero. We have

$$
\begin{aligned}
2 - C_{1n} - C_{2n} &= 0 \\
C_{1n} - \sigma C_{2n} &= 0 \\
-\frac{1}{2}C_{1n} - \frac{\sigma^2}{2}C_{2n} + \sigma^2 (B_{0n} + B_{1n} + B_{2n}) &= 0 \\
\frac{1}{6}(C_{1n} - \sigma^3 C_{2n}) - \sigma^2 B_{0n} + \sigma^3 B_{2n} &= 0 \\
-\frac{1}{24}(C_{1n} + \sigma^4 C_{2n}) + \frac{1}{2}(\sigma^2 B_{0n} + \sigma^4 B_{2n}) &= 0
\end{aligned} \qquad (4.236)
$$

and

$$L[y(x), h_n] = \left[\frac{1}{5!}(C_{1n} - \sigma^5 C_{2n}) + \frac{1}{3!}(\sigma^2 B_{0n} + \sigma^6 B_{2n})\right] h_n^5 y_{(x_n)}^{(v)} + \cdots \qquad (4.237)$$

where $h_{n+1} = \sigma h_n$.

Solving (4.236) for C's and B's we obtain

$$
\begin{aligned}
C_{1n} &= \frac{2\sigma}{(1+\sigma)}, & C_{2n} &= \frac{2}{(1+\sigma)} \\
B_{0n} &= \frac{1+\sigma-\sigma^2}{6\sigma(1+\sigma)}, & B_{1n} &= \frac{\sigma^3+4\sigma^2+4\sigma+1}{6\sigma^2(1+\sigma)} \\
B_{2n} &= \frac{\sigma^2+\sigma-1}{6\sigma^2(1+\sigma)}
\end{aligned} \qquad (4.238)
$$

The value $\sigma > 1$ gives more mesh points at small values of x while $\sigma < 1$ gives more mesh points at larger values of x. In most of the boundary value problems it is possible to know in advance the location of the boundary layer and the value of σ may be chosen suitably. In more general cases, in which the boundary layer is not known a priori we can compute the solution for some steplength h, repeat the calculation with another value of h and see whether the solution changes by more than an acceptable amount i.e., $|y_{i+1}-y_i| > \delta$, where δ is a prescribed limit.

In the boundary value problem (4.244i) the boundary layer exists near the right hand boundary $x = 1$. We choose $\sigma = 0.6$ and this gives more mesh points near $x = 1$. For $N = 8$, $\epsilon = 10^{-2}$, the solution values are shown in Figure 4.3(a).

The boundary layer in (4.244 ii) is near the point $x = -1$. We choose $\sigma = 1.2$ and this will give more mesh points near the left hand boundary. For $N = 100$, $\epsilon = 10^{-6}$ the solution values are shown in Figure 4.3(b).

We solve the boundary value problem (4.244 iii) over the interval $[-5, 5]$. The boundary layer exists near the origin. We choose a symmetric mesh about the origin. We take $\sigma > 1$ for the interval $[0, 5]$, while for the interval $[-5, 0]$ the reflection is used. The total number of mesh points in the interval $[-5, 5]$ are $2N+1$. For $N = 8$, $\epsilon = 1/24$, the resulting system of equations are solved with the Newton-Raphson iteration method. The solution values are shown in Figure 4.3(c).

We may conclude that the variable mesh method (4.243) is well suited for solving boundary layer problems. A priori knowledge of the location of he boundary layer is very helpful in producing accurate results with relatively little effort.

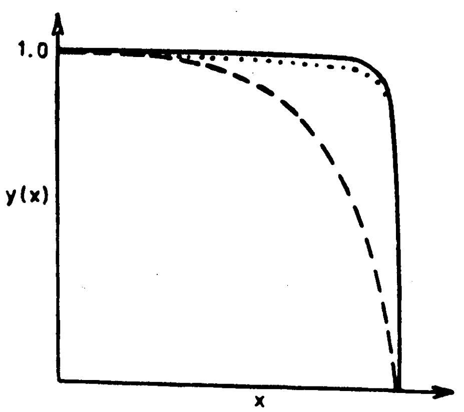

Fig. 4.3(a) Solution of $\epsilon y'' = y'$, $y(0) = 1$, $y(1) = 0$, $\epsilon = 10^{-2}$
——— exact, ... computed ($\sigma = 0.61$),
– – – computed ($\sigma = 1.0$)

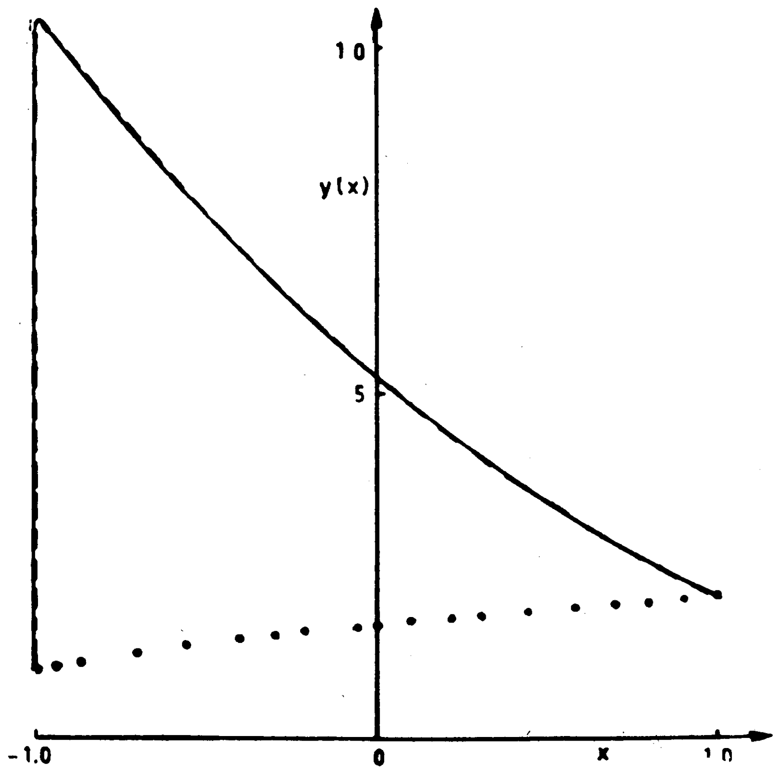

Fig. 4.3(b) Solution of $\epsilon y''+2y'+y = -3$, $y(-1) = 1$, $y(1) = 2$, $N = 100$, $\epsilon = 10^{-6}$
—— exact, ... computed ($\sigma = 1.0$), - - - computed ($\sigma = 1.2$)

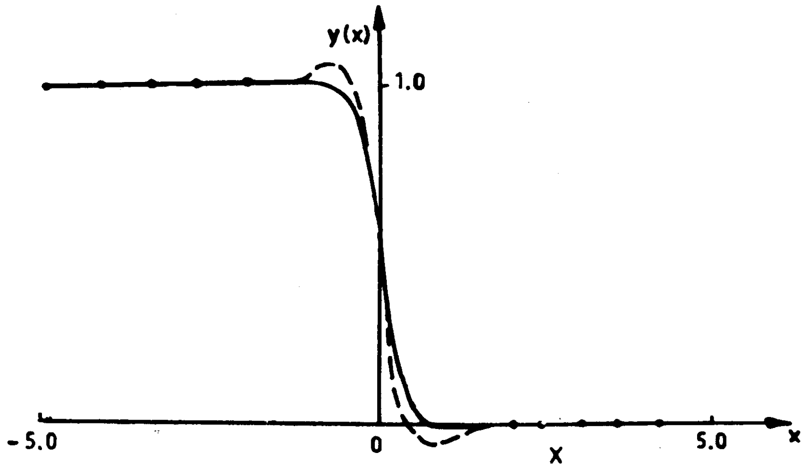

Fig. 4.3(c) Solution of $\epsilon y'' = (y-1/2)y'$, $y(-\infty) = 1$, $y(\infty) = 0$, $N = 8$, $\epsilon = 1/24$
—— exact, ... computed ($\sigma = 1.1$), - - - computed ($\sigma = 1.0$)

8. Determine the constants in the following relations

$$h^{-4}\delta^4 = D^4(1+a\delta^2+b\delta^4)+0(h^6)$$

$$hD = \mu\delta+a_1\Delta^3E^{-1}+(hD)^4(a_2+a_3\mu\delta+a_4\delta^4)+0(h^7)$$

(BIT 8(1968), 59)

9. Find the coefficients a and b in the operator formula

$$\delta^2+a\delta^4 = h^2D^2(1+b\delta^2)+0(h^8)$$ (BIT 8(1968), 138)

10. The differential equation $y''+x^2(y+1) = 0$ is given with boundary values $y = 0$ for $x = \pm 1$. Find approximate values of $y(0)$ and $y(1/2)$ using the second and fourth order difference schemes, with $h = 1/2$. (BIT 7(1967), 81)

11. Find difference approximations for the solution $y(x)$ of the boundary value problem

$$y''+8 \sin^2 \pi x\, y = 0,\ x \in [0, 1]$$

$$y(0) = y(1) = 1$$

with step length $h = 0.25$, using second and fourth order methods. Also find an approximate value for $y'(0)$. (BIT 8(1968), 246)

12. The difference scheme

$$\delta^2 y_n = h^2\left[\frac{7}{12}y''_n+\frac{5}{24}(y''_{n-r}+y''_{n+r})\right],\ r^2 = \frac{2}{5}$$

with *Approximation* I is used to replace the boundary value problem

$$y'' = f(x)y(x)+g(x)$$

$$y(a) = A,\ y(b) = B$$

by a system of linear equations

$$(-1+A_n)\,y_{n-1}+(2+B_n)\,y_n+(-1+C_n)\,y_{n+1} = D_n,\ 1 \leqslant n \leqslant N-1$$

$$y_0 = A,\ y_N = B$$

Determine A_n, B_n, C_n and D_n. Show that the principal part of the truncation error is given by

$$\left[\frac{19}{1512000}M_8+\frac{38-9\sqrt{10}}{86400}f_M M_6+\frac{(\sqrt{10}-2)}{3600}f'_M M_5\right]h^8$$

where $M_n = \max_{a\leqslant x\leqslant b} |y^{(n)}(x)|,\ f_M = \max_{a\leqslant x\leqslant b} |f(x)|$

and $f'_M = \max_{a\leqslant x\leqslant b} |f'(x)|$

13. The system of equations

$$(-1+A_n)\,y_{n-1}+(2+B_n)\,y_n+(-1+C_n)\,y_{n+1} = D_n,\ 1 \leqslant n \leqslant N$$

where

$$A_n = h^2\beta_0 f_{n-1}+h^4\gamma_0 F_{n-1}-P_{n-1}\left[1+\frac{13}{15}h^2 f_{n-1}+\frac{2}{15}h^3 f'_{n-1}\right]$$
$$-Q_{n-1}\left[1-\frac{h^2}{80}f_{n-1}\right]-P_{n+1}\left[1-\frac{h^2}{15}f_{n-1}\right],$$

$$B_n = h^2\beta_1 f_n+h^4\gamma_1 F_n-\frac{16}{15}h^2 f_n(P_{n-1}-P_{n+1})-\frac{7}{120}h^3 f'_n Q_n,$$

$$C_n = h^2\beta_2 f_{n+1}+h^4\gamma_2 F_{n+1}+P_{n-1}\left[1-\frac{h^2}{15}f_{n+1}\right]+Q_n\left[1-\frac{h^2}{80}f_{n+1}\right]$$
$$+P_{n+1}\left[1+\frac{13}{15}h^2 f_{n+1}-\frac{2}{15}h^3 f'_{n+1}\right],$$

$$D_n = -h^2(\beta_0 g_{n-1}+\beta_1 g_n+\beta_2 g_{n+1})-h^4(\gamma_0 G_{n-1}+\gamma_1 G_n+\gamma_2 G_{n+1})$$
$$+P_{n-1}\left[\frac{h^2}{15}(13g_{n-1}+16g_n+g_{n+1})+\frac{2}{15}h^3 g'_{n-1}\right]$$
$$-Q_n\left[\frac{h^2}{80}(g_{n-1}-g_{n+1})-\frac{7}{120}h^3 g'_n\right]$$
$$-P_{n+1}\left[\frac{h^2}{15}(g_{n-1}+16g_n+13g_{n+1})-\frac{2}{15}h^3 g'_{n+1}\right],$$

$$P_n = h^3\gamma_0 f'_n\left(1+\frac{h^2}{15}f_n\right)^{-1}$$

$$Q_n = h^3\gamma_1 f'_n\left(1+\frac{7}{60}h^2 f_n\right)^{-1}$$

$$f_i = f(x_i),\ g_i = g(x_i),\ F_i = f''_i+f_i^2$$
$$G_i = f_i g_i+g''_i$$

$$(\beta_0, \beta_1, \beta_2) = \frac{1}{15120}(660, 13800, 660)$$

$$(\gamma_0, \gamma_1, \gamma_2) = \frac{1}{15120}(-13, 626, -13)$$

which may also be written as

$$\mathbf{M\,y} = \mathbf{D}$$

represents difference replacement of the problem

$$y'' = f(x)y(x)+g(x)$$
$$y(0) = A,\ y(1) = B$$

(a) Show that $\mathbf{M} \rhd \mathbf{J}$ for $f(x) \geqslant 0$ on $[a, b]$

(b) Find out the condition for which $\mathbf{M}$ is irreducible

(c) Determine the error equation

$$\mathbf{M\,E} = \mathbf{T}$$

21. The difference equations

(i) $\left(1+h+\frac{5}{12}h^2f_1+\frac{h^3}{12}(f_1+f_1')\right)y_1+\left(-1+\frac{h^2}{12}f_2\right)y_2$

$$=-hA-\frac{h^3}{12}Af_1,$$

(ii) $\left(-1+\frac{h^2}{12}f_{n-1}\right)y_{n-1}+\left(2+\frac{10}{12}h^2f_n\right)y_n+$

$$\left(-1+\frac{h^2}{12}f_{n+1}\right)y_{n+1}=0,\ 2\leqslant n\leqslant N-1,$$

(iii) $\left(-1+\frac{h^2}{12}f_{N-1}\right)y_{N-1}+\left(1+h+\frac{5}{12}h^2f_N+\frac{h^3}{12}(f_N-f_N')\right)y_N$

$$=hB+\frac{h^3}{12}Bf_N,$$

for the boundary value problem

$$y''=f(x)y,\ x\in[0,1]$$
$$y'(0)-y(0)=A$$
$$y'(1)+y(1)=B$$

may be written in matrix form

$$\mathbf{My}=\mathbf{D}$$

(a) Determine the condition for which **M** is irreducible
(b) Find the error equation **ME** = **T**
(c) Show that

$$\|\mathbf{E}\|\leqslant\frac{1}{60}\left(M_5^*+\frac{3}{8}M_6\right)h^4$$

where $\overline{M}_5=\max_{0\leqslant x\leqslant h}|y^{(5)}(x)|$

$$\widetilde{M}_5=\max_{h(N-1)\leqslant x\leqslant 1}|y^{(5)}(x)|$$

$$M_5^*=\max(\overline{M}_5,\widetilde{M}_5)$$

and $M_6=\max_{0\leqslant x\leqslant 1}|y^{(6)}(x)|$.

22. Show that the differential equation

$$y''-y=2x,\ x\in[0,1]$$

may be replaced by the finite difference equations

$$\left(1-\frac{1}{12}h^2\right)y_{n-1}-\left(2+\frac{5}{6}h^2\right)y_n$$
$$+\left(1-\frac{1}{12}h^2\right)y_{n+1}=2h^2x_n+Cy_n$$

where $$C = \frac{1}{240}\delta^6 + \ldots$$

The boundary conditions are

$$y(0) = 0$$
$$y(1)+y'(1) = 1$$

Find the finit difference approximation to the boundary condition at $x = 1$ as accurately as that of the approximation to the differential equation; then solve the equation with $h = .25$.

23. Determine a difference approximation of the problem

$$[(1+x^2)y']' - y = x^2+1$$
$$y(-1) = y(1) = 0$$

Find approximate value of $y(0)$ using the steps $h = 1$ and $h = 0.5$ and also perform Richardson's extrapolation. (BIT 7(1967), 338)

24. Consider the boundary value problem

$$(p(x)y')' - f(x)y = g(x),\ x \in [a, b]$$
$$y'(a) - cy(a) = A$$
$$y'(b) + dy(b) = B$$

Using the difference equations

$$-p_{k-1/2}y_{k-1} + (p_{k-1/2} + p_{k+1/2} + h^2 f_k)y_k - p_{k+1/2}y_{k+1} = -h^2 g_k,\ 0 \leqslant k \leqslant N+1$$

as the difference replacement of the differential equation together with the following difference replacement for the boundary conditions:

$$\frac{y_1 - y_{-1}}{2h} - cy_0 = A,\ \frac{y_{N+2} - y_N}{2h} + dy_{N+1} = B$$

(i) write these difference equations as a system of order $N+2$;
(ii) if $f(x) > 0$ and the solution $y(x)$ is sufficiently smooth in an open interval containing $[a, b]$, show that for sufficiently small h

$$|\, y_k - y(x_k) \,| \leqslant 0(h^2),\ 0 \leqslant k \leqslant N+1$$

25. The solution of the two-point boundary value problem

$$y'' = f(x, y),\ x \in [a, b]$$

where $y(a)$ and $y(b)$ are given, is usually found by use of a recurrence relation of the form

$$-y_{n-1} + 2y_n - y_{n+1} + h^2 f_n = 0,\ 1 \leqslant n \leqslant N-1$$

Neglecting round-off errors, show that the bound for the truncation errors T_m is given by

$$|\, T_m \,| \leqslant \frac{1}{12} h^4 \sum_{n=1}^{N-1} j_{mn}$$

where j_{mn} are the elements of the inverse of $N-1\times N-1$ matrix

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

26. The solution of the mixed boundary value probem

$$y'' = f(x, y),$$
$$y'(a)-cy(a) = A,$$
$$y'(b)+dy(b) = B$$

is obtained by solving the system of nonlinear equations

$$\left[1+hc\left(1+\frac{h^2}{12}\bar{q}_1\right)\right]y_1-y_2$$
$$+\frac{h^2}{12}(5\bar{f}_1+\bar{f}_2)+hA\left(1+\frac{h^2}{12}\bar{q}_1\right)+\frac{h^3}{12}\bar{p}_1 = 0,$$
$$-y_{n-1}+2y_n-y_{n+1}+\frac{h^2}{12}(\bar{f}_{n-1}+10\bar{f}_n+\bar{f}_{n+1}) = 0,$$
$$2 \ldots n \ldots N-1,$$
$$-y_{N-1}+\left[1+hd\left(1+\frac{h^2}{12}\bar{q}_N\right)\right]y_N$$
$$+\frac{h^2}{12}(\bar{f}_{N-1}+5\bar{f}_N)-hB\left(1+\frac{h^2}{12}\bar{q}_N\right)-\frac{h^3}{12}\bar{p}_N = 0$$

where $\bar{f}_m = f(x_m, y_m)$, $\bar{p}_m = f_x(x_m, y_m)$

and $\bar{q}_m = f_y(x_m, y_m)$

Find the error equation **ME = T**.

27. The function $y(x)$ is determined through

$$y'''-x^2y = 0,\ y(0) = \frac{1}{2},$$
$$y'(0) = 0,\ y(1) = 1$$

Find $y(1/2)$ with the help of the difference method. Use $h = 1/4$.

28. Solve the boundary value problem

$$y^{iv} = 1,\ y(0) = y'(0) = y(1) = y'(1) = 0$$

with the help of the second order difference method with $h = .25$. Apply the elimination method for solving the system of equations.

29. Solve the fourth order differential equation

$$y^{iv}-y'''+y = x^2$$

subject to the boundary conditions

$$y(0) = 0,\ y'(0) = 0$$
$$y(1) = 2,\ y'(1) = 0$$

with $h = .2$.

30. Find the error equation $\mathbf{ME} = \mathbf{T}$ of the system of linear equations (4.189).

31. Let M_0 be the five band $N+1 \times N+1$ matrix defined by

$$\begin{bmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{bmatrix}$$

(a) Determine $M_0^{-1} = (m_{ij})$ explicitly and show that it is a symmetric and monotone matrix

(b) Prove that

$$\| \mathbf{E} \| \leqslant \frac{(b-a)^4}{384} h^2 M_6$$

where $$M_6 = \max_{a \leqslant x \leqslant b} | y^{(6)}(x) |$$

32. Obtain approximate values of the smallest characteristic number Λ_1 for the problem

$$y'' + \Lambda(1+x^2)y = 0,\ y(\pm 1) = 0$$

Use the second order method with N successively taken as $N = 1$ and 2, $h = 1/N$.

33. Consider the homogeneous boundary value problem

$$y'' + \Lambda\, y = 0,$$
$$y(0) = y(1) = 0$$

(a) Show that the application of the fourth order Numerov method leads to the system

$$\left[\mathbf{J} - \frac{\lambda}{1 + \frac{\lambda}{12}} \mathbf{I} \right] \mathbf{y} = 0$$

where $\lambda = h^2 \Lambda$.

(b) The approximations to the eigenvalues by the second and the fourth order methods are given by $\frac{2}{h^2}(1 - \cos n\pi h)$ and

$\frac{12}{h^2} \frac{1-\cos n\pi h}{5+\cos n\pi h}$, $1 \leqslant n \leqslant N$ respectively, where $h = 1/(N+1)$.

(c) Noticing that $\Lambda_n = n^2\pi^2$, show that the relative error

$$\frac{\Lambda_n - h^{-2}\lambda_n}{\Lambda_n}$$

for the second and the fourth order methods is given by $\Lambda_n h^2/12$ and $\Lambda_n^2 h^4/240$, respectively when terms of higher order in h are neglected.

34. The sixth order difference scheme based on the four-point Lobatto quadrature rule with the *Approximation* I is applied to the problem $y''+\Lambda y = 0$, $y(0) = y(1) = 0$. Establish:

(a) that the system of equations is obtained as

$$\left[\mathbf{J} - \frac{\left(1-\frac{\lambda}{60}\right)}{1-\frac{\lambda}{15}+\frac{\lambda^2}{360}} \mathbf{I} \right] \mathbf{y} = 0$$

where $\lambda = h^2 \Lambda$.

(b) the approximation of the egienvalue is given by
$(4-\cos n\pi h)\lambda^2 - 12(13+2\cos n\pi h)\lambda + 360(1-\cos n\pi h) = 0$

(c) the relative error is found as

$$\frac{\Lambda_n - \lambda_n h^{-2}}{\Lambda_n} = \frac{191}{302400} \Lambda_n^3 h^6 + 0(h^8)$$

35. Apply the second and fourth order difference methods to the problem

$$y''+\Lambda y = 0$$
$$y'(0) = y(0),\ y(1) = 0$$

Write the characteristic equation in the form
$| \mathbf{A} - \lambda \mathbf{B} | = 0$. Find $\mathbf{A}$ and $\mathbf{B}$, where $h = 1/N$.